DETERMINING MORPHOMETRIC AND HYDRAULIC CHANNEL CHARACTERISTICS IN INTEGRATION OF ST. VENANT EQUATIONS

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Meteorologiya i Gidrologiya, No. 8, pp. 71-80, 1976

UDC 556.013

Certain special features in solving inverse problems for St. Venant equations are considered. Results are given for computation of a nonsteady regime for the reach of the Volga River downstream of the XXII CPSU Congress Hydroelectric Power Plant, using reduced morphometric and hydraulic channel characteristics.

For large rivers with regulated flows and great variability in hydraulic and morphometric channel characteristics along their lengths, simplified methods of computing irregular water motion often seem ineffective. In this connection, numerous attempts have been made to use St. Venant equations for this purpose. The overwhelming majority of the studies have been directed toward developing effective finite-difference schemes for numerical integration of a given system of equations. These studies have shown that even with the use of the most effective difference schemes and very detailed measurements of morphometric (channel width, cross-sectional area) and hydraulic (modulus of flow, roughness coefficient) channel characteristics, the accuracy of the computations of irregular motion seems inadequate for real rivers. One of the reasons for this low accuracy is the large error in definition of channel characteristics (especially the modulus of water flow). On the other hand, inadequacy of the model considered and of the actual processes occurring in the channel has an effect.

To increase accuracy in computing irregular motions, semiempirical correction of the original information [1] was usually carried out by correlation of calculated and actual discharges and water levels. This is extremely cumbersome and seldom leads to satisfactory results. Moreover, instances are not unusual in which, due to limited observational data, it is generally not possible to give initial approximations of the channel characteristics to be considered.

A number of articles [3-5] have been devoted to developing more general objective methods of determining morphometric and hydraulic channel characteristics according to data from observations on an irregular water regime, based on solving inverse problems for St. Venant equations. However, in these articles there is scarcely any consideration of problems relating to the use of reduced channel characteristics during numerical integration of St. Venant equations.

Let us consider, using the example of the Volga River below the XXII CPSU Congress Hydroelectric Plant, some special features of determining morphometric and hydraulic channel characteristics in the case of a discrete representation of water levels along the length of the river and water discharges in the initial and final sites, and also the possibilities of using reduced characteristics in solving a direct problem for St. Venant equations.

Let us use a system of St. Venant equations having the following notation:

$$\frac{1}{g}\frac{\partial Q}{\partial t}+\frac{1}{g}\frac{\partial\left(\frac{Q^2}{F}\right)}{\partial x}+F\frac{Q|Q|}{K^2}+F\frac{\partial H}{\partial x}=0, \tag{1}$$

$$\frac{\partial Q}{\partial x}+\frac{\partial F}{\partial t}=q. \tag{2}$$

where H(x,t) is the water level at point x at time t; Q(x,t) is the water discharge; F(x,H) is the corss-sectional area; K(x,H) is the modulus of discharge (passing capacity of channel); q(x,t) is the lateral flow (run-off) of water per unit of length; g is free fall acceleration.

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The main difficulties in integrating equations (1) and (2) relate to the representation of functions F(x,H) and K(x,H). Values of area and modulus of water discharge measured at individual sites along the length of the river seem nonrepresentative for the entire length of the river, which, as has been noted above, leads to a large error in computing discharges and water levels.

Following [3], we reduce differential equations (1) and (2) to an integral form, ignoring the inertional terms in (1) and replacing the modulus of water discharge with a Chezy-Manning relationship. In accordance with available information about discharges and water levels, the integral equations are most conveniently represented in the following form:

$$\int_0^L \{F[x, H(x, T_{l+1})] - F[x, H(x, T_l)]\}\, dx = \int_{T_l}^{T_{l+1}} [Q(0, t) - Q(L, t) - \Delta Q(t)]\, dt, \tag{3}$$

$$\int_0^L \frac{Q|Q|}{F^2 (F/B^{4/3})} n^2 (x, H)\, dx = H(0, t) - H(L, t), \tag{4}$$

where $\Delta Q(t) = \int_0^L q dx = (1-k)\, Q(0, t)$ are losses on the computed portion of the river, k is the ratio of water volumes in the closed and inlet sections, Q(0,t), H(0,t) are discharges and water levels in the inlet section, Q(L,t), H(L,t) are discharges and water levels in the closed section, B = dF/dH is the channel width, n(x,H) is the Chezy-Manning roughness coefficient.

For a solution of equation (4), the distribution of water discharges along the length must be determined from a continuity equation after equation (3) has been solved:

$$Q_p(x, t) = Q(0, t) - \int_0^x \frac{\partial F}{\partial t}\, d\eta. \tag{5}$$

Since the water discharges calculated according to relationship (5) might differ from the actual discharges, a correction was introduced, proportional to the discrepancy between the computed and actual discharges for the closed section.

$$Q_p'(x, t) = Q_p(x, t) - [Q_p(L, t) - Q_p(L, t)]\, \frac{x}{L}.$$

We shall seek the solution of equations (3) and (4) in the form of an expansion:

$$F(x, H) = \sum_{s=0}^{n} D_s(x)\, \psi_s(H), \tag{6}$$

$$n^2(x, H) = \sum_{s=0}^{m_1} l_s(x)\, \psi_s(H), \tag{7}$$

where $\psi_s(H)$ is a complete orthogonal system of Chebyshev polynomials.

Substituting (6) into (3), we obtain the expression

$$\sum_{s=0}^{m} \int_0^L D_s(x) \{\psi_s[H(x, T_{l+1})] - \psi_s[H(x, T_l)]\}\, dx = \tag{8}$$

$$= \int_{T_i}^{T_{i+1}} \{Q(0, t) - Q(L, t) - \Delta Q(t)\}\, dt. \tag{8}$$

In equation (8) $\psi_0[H(x, T_{i+1})] = \psi_0[H(x, T_i)]$, which makes it practically impossible to determine coefficients $D_0(x)$. We shall therefore determine function $B(x,H)$ from (8), not including coefficients $D_0(x)$:

$$B(x, H) = \frac{dF(x, H)}{dH} = \sum_{s=1}^{m} D_s(x) \frac{d\psi_s(H)}{dH}. \tag{9}$$

We shall determine the defective zero coefficients $D_0(x)$ by integrating equation (9) over H and substituting expression (6) for $F(x,H)$:

$$D_0(x) = F(x, H_m) - \sum_{s=1}^{m} D_s(x)\, \psi_s(H_m). \tag{10}$$

To solve equation (10) it is necessary to represent values of the cross-sectional area along the length for certain levels H_m. It would be convenient to represent the zero area of the cross section as corresponding to the elevation of the thalweg (H_0) of the river channel. However, this condition can be used only when the amplitude of the actually measured levels of water used in solving equation (3) corresponds with the section $[H_0, H_{max}]$, since in this case normalization of H in the [0,1] section does not violate the condition of representing $F(x,H)$ and $B(x,H)$ according to a complete and orthogonal system of polynomials. In the opposite case, the region of determination of the normalized function will not coincide with the region of occurrence of a system of orthogonal approximating polynomials. As numerical experiments have shown, violation of this condition can lead to a significant distortion of computed functions.

For actual rivers, measured values of $F(x,H_m)$ are not always available; therefore this function must be determined indirectly, by assigning the elevation of the thalweg and introducing several assumptions about the shape of the channel below the elevation H_m. Specifically, a parabolic channgel shape was assigned for the section of the Volga considered. Then the area will be equal to:

$$F(x, H_m) = \frac{2}{3} B(x, H_m) |H_m(x) - H_0(x)|. \tag{11}$$

We shall seek the coefficients of expansion in (6) and (7) for several points along the length, limited to sites of water level change. Then, substituting the integral of the sum into (8) and excluding zero coefficients of expansion, we obtain a system of linear algebraic equations

$$W\vec{D} = \vec{U}, \tag{12}$$

here W is the matrix of the series $z \times (mN)$, the elements of which are equal to

$$w_{ij} = \begin{cases} \dfrac{\psi_s[H(x_\beta, T_{i+1})] - \psi_s[H(x_\beta, T_i)]}{2} (x_{\beta+1} - x_\beta) & \text{at } \beta = 1 \\ \dfrac{\psi_s[H(x_\beta, T_{i+1})] - \psi_s[H(x_\beta, T_i)]}{2} (x_{\beta+1} - x_{\beta-1}) & \text{at } 1 < \beta < N \\ \dfrac{\psi_s[H(x_\beta, T_{i+1})] - \psi_s[H(x_\beta, T_i)]}{2} (x_\beta - x_{\beta-1}) & \text{at } \beta = N \end{cases}$$

$$(i = 1, 2, \ldots, z;\ j = 1, 2, \ldots, mN),$$

where z is the number of moments of time whose data are included in the computation (not less than mN); N is the number of water level observation points; $\beta = \mathrm{ent}[(j + m - 1)/m]$; $s = j - (\beta - 1)m$; $\vec{D}$ is the vector of the desired coefficients of series mN; $\vec{U}$ is the vector of series z with elements

$$u_i = \int_{T_i}^{T_{i+1}} [Q(0, t) - Q(L, t) - \Delta Q(t)]\, dt. \tag{13}$$

Due to errors in the original data, matrix W and vector $\vec{U}$ will include a certain error. This circumstance, as a rule, makes it impossible to determine vector $\vec{D}$ reliably, since the solution of system (12) is very sensitive to error, i.e., is unstable.

To solve this problem, let us use an A. N. Tikhonov functional of the first order [2]

$$R^{\alpha}[\vec{D}_{\delta}^{\alpha}, \vec{U}_{\delta}] = \| \tilde{W}\vec{D}_{\delta}^{\alpha} - \vec{U}_{\delta} \|^2 + \alpha \| \vec{G}_{\delta}^{\alpha} \|_1^2.$$

Here $\tilde{W}, \tilde{U}_{\delta}$ are, respectively, matrix W and vector $\vec{U}$, assigned with an error δ; $\|\cdot\|$ here and subsequently is a spherical norm,

$$\| \vec{G}_{\delta}^{\alpha} \|_1^2 = \sum_{s=1}^{m} \int_0^L \left\{ (\vec{D}_s^{\alpha})^2 + \left(\frac{d\vec{D}_s^{\alpha}}{dx} \right)^2 \right\} dx,$$

α is the parameter of regularization.

The regularizing algorithm in the given problem will be a process of minimization of the functional R^{α}, including selection of parameter α_{opt} according to an inequality from [2]

$$\alpha_{\psi} \leq \alpha_{opt} \leq \alpha_s \leq \alpha_{\varphi}, \tag{14}$$

where α_{ψ} corresponds to the local maximum of the function

$$\psi_i(\alpha) = \frac{\| \tilde{W}\alpha \frac{d\vec{D}_i^{\alpha}}{d\alpha} - (\tilde{W}\vec{D}_i^{\alpha} - \vec{U}_{\delta}) \|^2}{\| \tilde{W}\vec{D}_i^{\alpha} - \vec{U}_{\delta} \|^2},$$

α_s and α_{φ} correspond to local minimums of functions

$$s_i(\alpha) = \| \vec{D}_i^{\alpha_{i+1}} - \vec{D}_i^{\alpha_i} \|^2 \text{ and } \varphi_i(\alpha) = \left\| \alpha \frac{d\vec{D}_i^{\alpha}}{d\alpha} \right\|^2, \quad \alpha_i = \alpha_0 Д^i$$

(Д is the increment of change of α).

The sought vector D_{δ}^{α}, minimizing R^{α}, represents the solution of a system of linear algebraic equations

$$(\tilde{W}^* \tilde{W} + \alpha C) \vec{D}_{\delta}^{\alpha} = \tilde{W}^* \vec{U}_{\delta} \tag{15}$$

with consideration of the inequality of (14).

In equation (15) $\tilde{W}^*$ is a matrix transposing toward $\tilde{W}$; C is a three-diagonal quadratic matrix of the series mN, the elements of which are equal to

$$c_{ij} = \begin{cases} \dfrac{x_{\beta} - x_{\beta-1}}{2} + \dfrac{1}{x_{\beta} - x_{\beta-1}} + \dfrac{x_{\beta+1} - x_{\beta}}{2} + \\ \quad + \dfrac{1}{x_{\beta+1} - x_{\beta}} & \text{at } i=j,\ 1<\beta<N \\ \dfrac{x_{\beta+1} - x_{\beta}}{2} + \dfrac{1}{x_{\beta+1} - x_{\beta}} & \text{at } i=j,\ \beta=1 \\ \dfrac{x_{\beta} - x_{\beta-1}}{2} + \dfrac{1}{x_{\beta} - x_{\beta-1}} & \text{at } i=j,\ \beta=N \\ -\dfrac{1}{x_{\beta+1} - x_{\beta}} & \text{at } j=i+m \\ -\dfrac{1}{x_{\beta} - x_{\beta-1}} & \text{at } j=i-m \\ 0 \end{cases} \tag{16}$$

in the remaining cases

$$(i=1, 2, \ldots, mN;\ j=1, 2, \ldots, mN)$$

$$\beta = \mathrm{ent}\left[\frac{i+m-1}{m}\right].$$

Having carried out analogous operations for equation (4), we can construct a stable system for determining coefficients of expansion of function $n^2(x,H)$:

$$(\tilde{M}^* \tilde{M} + \alpha C) \vec{P}^\alpha_\delta = \tilde{M}^* \vec{\Delta}_\delta, \qquad (17)$$

where $\vec{P}^\alpha_\delta$ is the vector of the unknown coefficients; $\tilde{M}$ is the matrix of the series $z \times (m_1+1)N$, the elements of which are equal to

$$m_{ij} = \begin{cases} \nu_{ij} \dfrac{\psi_s [H(x_\beta, T_i)]}{2} (x_{\beta+1} - x_\beta) & \text{at } \beta = 1 \\ \nu_{ij} \dfrac{\psi_s [H(x_\beta, T_i)]}{2} (x_\beta - x_{\beta-1}) & \text{at } \beta = N \\ \nu_{ij} \dfrac{\psi_s [H(x_\beta, T_i)]}{2} (x_{\beta+1} - x_{\beta-1}) & \end{cases}$$

in the remaining cases

$$(i = 1, 2, \ldots, z;\ j = 1, 2, \ldots, (m_1+1)N)$$

$$\beta = \text{ent}\left[\frac{j+m_1}{m_1+1}\right], \quad s = j - \beta - (\beta - 1)\, m_1,$$

$$\nu_{ij} = \frac{[Q_p'(x_\beta, T_i)]^2 \, [B(x_\beta, T_i)]^{4/3}}{[F(x_\beta, T_i)]^{10/3}},$$

C is the three-diagonal quadratic matrix of the series $(m_1 + 1)N$ similar to (16), only with consideration of zero degrees of approximating polynomials; $\vec{\Delta}_\delta$ is the vector of the series z with the elements

$$\delta_i = H(0, T_i) - H(L, T_i),$$

functions $F(x,t)$ and $B(x,t)$ are determined in accordance with (6) and (7).

In the portion of the Volga considered, which extends for 448 km, there were 9 gaging stations in which water level observations were carried out (XXII CPSU Congress Hydroelectric Power Plant, Volgograd, Krasnoarmeisk, Svetlyi Yar, Kamennyi Yar, Chernyi Yar, Enotaevka, Seroglazovka, Verkhnelebyazh'e). In this connection, as a result of solving systems (15) and (17), values of coefficients of expansion D_s and P_s were found for each of these gaging stations. The total number of coefficients depends on the number of polynomials $(m + m_1 + 2)$ taken for functions $F(x_i,H)$ and $n^2(x_i,H)$.

The optimal number of polynomials approximating function $F(x_i,H)$ was determined starting from the minimum of the following functional:

$$\Phi = \sum_{i=1}^{z} [Q(L, T_i) - Q_p(L, T_i)]^2. \qquad (18)$$

Roughness coefficients change slightly with change of water level. It therefore seemed possible to use a small number of polynomials ($m_1 = 2$). A further increase of the number of polynomials did not lead to any significant change in the reduced functions.

Figure 1 shows reduced cross-sectional areas and roughness coefficients at m = 6 and $m_1 = 2$. The computed channel characteristics are rather stable functions, not varying appreciably from year to year. The substantial deviations in 1966 are explained, first, by a large error in the original data, and, second, by the complicated tidal structure of the Volga River channel, and the irregular, dense network of constant and intermittent watercourses. The degree of flooding of the floodplain varies greatly from year to year, which can lead to a different sort of averaging of channel characteristics.

USE OF COMPUTED CHANNEL CHARACTERISTICS IN CALCULATING UNSTABLE REGIME

For numerical integration of equations (1) and (2) we shall use the Lakes-Vendroff explicit scheme. As is shown in [4], this scheme is well suited to uneven original channel characteristics and at the same time has a relatively slight flattening.

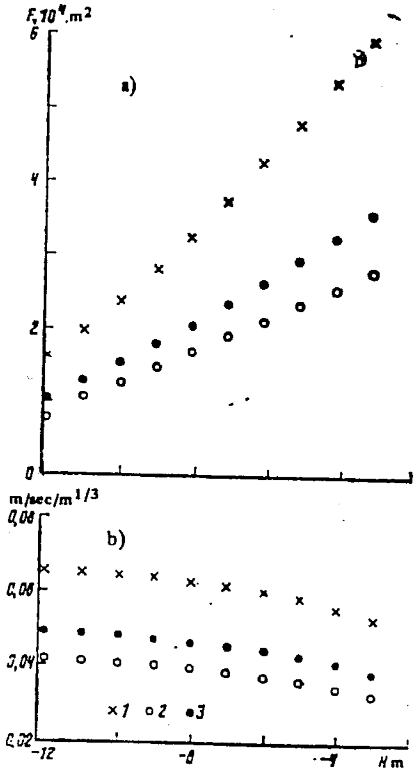

Fig. 1

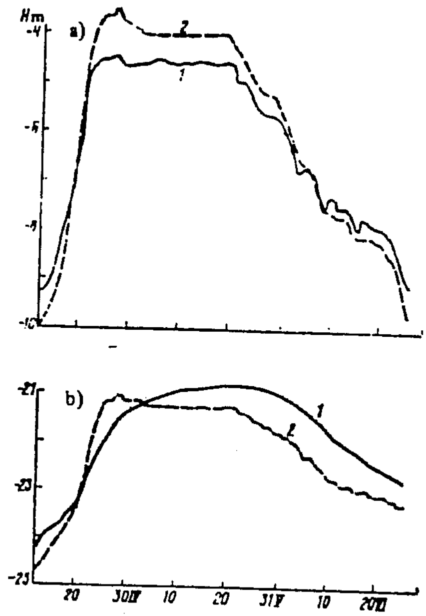

Fig. 2

Fig. 1. Reduced cross-sectional areas (a) and roughness coefficients (b) for Volga River at Krasnoarmeisk gaging station, based on data from 1966 (1); 1969 (2); 1970 (3).

Fig. 2. Actual (1) and computed (2) water levels of the Volga River during 1970 for Krasnoarmeisk (43 km from power plant) (a) and Verkhnelebyazh'e (448 km from power plant) (b).

In accordance with the Lakes-Vendroff scheme, in odd-numbered increments we replace the partial derivatives over x and t with differential relationships of the form

$$\frac{\partial f}{\partial t} = \frac{f_i^{2j+1} - \frac{1}{2}\left(f_{i+1}^{2j} + f_{i-1}^{2j}\right)}{\Delta t}, \quad \frac{\partial f}{\partial x} = \frac{f_{i+1}^{2j} - f_{i-1}^{2j}}{2\,\Delta x}, \tag{19}$$

and in even-numbered ones,

$$\frac{\partial f}{\partial t} = \frac{f_i^{2(j+1)} - f_i^{2j}}{2\,\Delta t}, \quad \frac{\partial f}{\partial x} = \frac{f_{i+1}^{2j+1} - f_{i-1}^{2j+1}}{2\,\Delta x}, \tag{20}$$

where Δt and Δx are increments of integration over time and distance, respectively, i,j are numbers of points of the differential grid along axes x and t, respectively.

For odd-numbered moments of time (2j + 1) the coefficients and right-hand parts of equations (1) and (2) were approximated in the following manner:

$$f(x, t) = \frac{1}{2}\left(f_{i+1}^{2j} + f_{i-1}^{2j}\right), \tag{21}$$

and for even-numbered moments of time (2j + 2),

$$f(x, t) = f_i^{2j}. \tag{22}$$

Substituting (19)-(22) into equations (1) and (2), we construct a system of differential equations for determining the discharges and cross-sectional areas at all internal points of the grid region to be considered. For computation of the boundary values of the unknown functions, the characteristic equation was used:

$$\frac{\partial Q}{\partial x}+\frac{\partial F}{\partial t}-q\pm\sqrt{\frac{B}{gF}}\left\{\frac{\partial Q}{\partial t}+\frac{\partial\left(\frac{Q^2}{F}\right)}{\partial x}+gF\cdot\frac{\partial H}{\partial x}+gF\frac{Q|Q|}{K^2}\right\}=0$$

and the following boundary conditions were used:

$$Q(0,t)=f_1(t),\quad Q(L,t)=f_2[F(L,t)].$$

The left boundary coincided with the XXII CPSU Congress Hydroelectric Power Plant site, the right boundary with the Verkhnelebyazh'e site.

Observations of lateral affluent (outflow) are not made on the segment considered; the magnitude of q was therefore determined by computational means, allowing for the degree of flooding of the floodplain:

$$q(x,t)=(1-k)\,f_1(t)\,\xi(x,t)\,abs\left[\int_0^L\xi(x,t)\,dx\right].$$

$$\xi(x,t)=\frac{H(x,t)-H_n(x)}{H_s(x)-H_n(x)},$$

where $H_n(x), H_s(x)$ are water levels characterizing, respectively, escape of water onto the floodplain, and its complete flooding.

The use of morphometric and hydraulic channel characteristics reduced by the above means is made difficult for two reasons: first, it is necessary to carry out integration of these characteristics into nodes of the differential grid, since for actual rivers the distances between sites for which these functions are reduced are significantly larger than the increment of integration over length; second, computation of roughness coefficients and cross-sectional areas according to relationships (6) and (7) can be carried out reliably only in the range of levels used in solving the inverse problem.

In the case considered a linear integration of levels and hydraulic drag calculated for nine gaging stations was employed. Because of the great variability of cross-sectional areas from station to station, interpolation according to absolute area values seemed practically impossible. The range of variation of areas for one site may be too large (small) for another, which requires a significant extrapolation of functions H(F) and K(F). As has been noted above, such extrapolation is difficult.

Due to this circumstance, interpolation was carried out, not according to absolute values of the areas at points of the differential grid (F_i^j), but by the use of relative magnitudes corresponding to the increment of areas over certain of their initial distributions:

$$F_k^j=F_k^0+F_i^j-F_i^0.$$

where F_k^0 and F_i^0 are values of areas of the cross section at the initial moment of time, respectively, for the k-th station and the i-th point of the differential grid. Such a technique made it possible to narrow down the region of scanning of water levels and moduli of discharges very greatly, and to reduce to a minimum the extrapolation of functions H(F) and K(F). Negligible extrapolation was carried out, with allowance for the gradient of levels and moduli at the extremities of the considered observations of portions of curves H(F) and K(F); for example, the following expression may be written for H

$$H_x=H_m-\left.\frac{\partial H}{\partial F}\right|_{H=H_m}dF,$$

where H_m is the water level corresponding to the maximum (minimum) water level included in the sampling during solution of the inverse problem.

At the initial moment of time, the only water levels assigned are at gages of sites between which linear interpolation was carried out in the nodes of the differential grid. The initial water discharges were computed according to a simplified motion equation, with allowance for a distribution of cross-sectional areas which was not uniform in length:

$$Q=\sqrt{\left(-\frac{\partial H}{\partial x}\right)\Big/\left(\frac{1}{K^2}-\frac{1}{gF^2}\frac{\partial F}{\partial x}\right)}.$$

Computations of the transformation of run-offs of the XXII CPSU Congress Volga Hydroelectric Power Plant have shown that the solution obtained was stable and agreed rather well with actual water levels at lower-lying stations. Fig. 2 shows actual and computed (at $\Delta x = 11.2$ km, $\Delta t = 5$ min) water levels for the two sites. It can be seen that the results shown in the literature [1,6] for computations of irregular motion according to St. Venant equations with the use of morphometric and hydraulic channel characteristics, obtained by observations on a standard hydrometric grid, are markedly inferior in accuracy to the computations shown in Fig. 2.

From a practical point of view, however, such an accuracy is still not sufficient. Errors are in this instance caused mainly by the presence of an irregularly shaped floodplain, which leads to a substantial violation of the hypothesis of uniformity of motion, limitedness of information about water discharges, and also the use of a "fictitious" thalweg, which given the sharply expressed serrated nature of the longitudinal profile of the river channel, is ambiguously determined.

A further increase in accuracy could obviously be attained by differentiated determination of channel characteristics for the lower and upper (during escape of water onto the floodplain) water levels, or by correcting certain of obtained coefficients of expansion by means of optimalization methods, by analogy with [3].

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15 March 1976

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